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# Simultaneous eigenvectors for general Abelian hypergroups and restrictions imposed by C*QFT on the modular–conformal fusion algebra

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**Abstract.** Starting from the commutativity and associativity properties which characterize an abstract Abelian hypergroup, a system of simultaneous, dually eigenvector–eigenvalue quantities is constructed, in terms of the matrices defining the hypergroup and of a set of generators of an elementary algebra, which possesses trivial representations. It is observed that the particular (symmetric) expressions of Rehren for such simultaneous eigenvectors—derived in the context of algebraic QFT—actually render supplementary restrictions to Verlinde's algebra corresponding to the modular invariance of the partition function of two-dimensional conformal field theory.

## 1. Introduction

As has been shown during the last couple of years (for a good review on the subject see [1]), the most important relations which had been considered (by many) to be peculiar for two-dimensional (2D) conformal field theories can actually be derived as properties of the superselection sectors of a generic quantum field theory. This constitutes a remarkable example of the power which is embodied in the structure of the so-called C* or algebraic QFT. The last and very impressive successes of this formalism have been, on the one hand, the construction of explicit formulae for the endomorphisms and for the associated statistics operators of the non-Abelian sectors corresponding to the conformal Ising field theory [2] and, on the other, the derivation carried out by Rehren [3]—in the general C* framework—of E Verlinde's algebraic relations [4] expressing the monodromies and modular invariance of the partition function of 2D conformal field theories. That not only the general relations of CFT but also concrete structural elements of such specific nature can be derived from generic algebraic QFT in two dimensions—without using at any instance the concepts of conformal or modular invariance—is, to say it plainly, rather mysterious.

In this paper I shall investigate in some detail several questions posed, explicitly or implicitly, by Rehren's derivation of the Verlinde algebra. In particular, I shall touch upon the following two points. (i) How far can one go into the explicit construction of a system of simultaneous eigenvector–eigenvalues for a generic Abelian hypergroup, in terms of the matrices which define it? This question has been explicitly posed

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in [3]. It will be the subject of section 2. (ii) Is the algebra generated by Rehren's operators exactly the same as the Verlinde one, corresponding to the modular invariance of 2D CFT? In particular, the $S$ operators of Rehren possess a supplementary symmetry, which is, indeed, very useful in order to derive the modular-conformal algebra but not compulsory in order to satisfy the eigenvector-eigenvalue equations of the hypergroup. Actually, in [3] (and also in [1]) attention has been paid only to the fact that the algebraic relations of Verlinde are conveniently reproduced, but not to the question of the possible appearance of *additional* relations, specifically corresponding to the particular definitions of the $S$ and $T$ operators adopted in [3]. This is investigated in section 3. Finally, in section 4 a brief summary of the original results obtained in this paper is given.

## 2. Construction of a system of simultaneous eigenvector–eigenvalues for a generic Abelian hypergroup

Proceeding with the first point of the introduction, let us start from the equations which define a (finite) general Abelian hypergroup:

$$\sum_k N_{ij}^k N_{lk}^m = \sum_k N_{li}^k N_{kj}^m$$
$$N_{ij}^k = N_{ji}^k = N_{i\bar{k}}^{\bar{j}} = N_{\bar{i}\bar{j}}^{\bar{k}} \qquad N_{0j}^k = \delta_j^k. \tag{1}$$

The problem is to determine a set of simultaneous, dually eigenvector–eigenvalue quantities—which we shall call $Y_{ij}$—for the given hypergroup, e.g. such that the following equation is verified for all values of the free indices:

$$\sum_k N_{ij}^k Y_{hk} = \frac{1}{\alpha_h} Y_{hi} Y_{hj} \tag{2}$$

here we put $\alpha^{-1}$ (instead of $\alpha$) for later convenience. That this is in fact the eigenvalue equation for the matrix hypergroup, and that the $Y$'s are dually eigenvectors and eigenvalues is easily appreciated by writing the preceding equation under the compact, matrix-vector form

$$N_i Y_h = \frac{1}{\alpha_h} Y_{hi} Y_h \tag{3}$$

where the matrices $N_i$ and the vectors $Y_h$ are defined by

$$(N_i)_j^k = N_{ij}^k \qquad (Y_h)_j = Y_{hj} \tag{4}$$

respectively.

Now, we observe that equation (2) is an immediate consequence of equation (1) provided we construct the $Y_{ij}$ in terms of the $N_{ij}^k$'s and of a set of elements $e_i$, $e^i$, satisfying

$$e_i \cdot e^j = \delta_i^j \tag{5}$$

($\delta_i^j$ is the Kronecker symbol and the dot should mean, in principle, an abstract algebraic operation, non-commutative in general but with the usual bilinear and associative properties), in the following way:

$$Y_{ij} = \alpha_i \sum_{lm} N_{jl}^m e^l \cdot e_m. \tag{6}$$

In fact, from the double contraction of equation (1)

$$\sum_k N_{ij}^k \sum_{lm} N_{kl}^m e^l \cdot e_m = \sum_k \sum_l N_{il}^k e^l \cdot \sum_m N_{kj}^m e_m$$
$$= \sum_{lk} N_{il}^k e^l \cdot e_k \cdot \sum_{hm} N_{hj}^m e^h \cdot e_m \tag{7}$$

equation (2) follows immediately. Notice that the only properties which we have used in order to obtain this (very general) result are the commutativity and associativity of the $N_{ij}^k$. Summing up: every abstract (finite) Abelian hypergroup—whatever the actual meaning or additional properties of the $N_{ij}^k$—possesses a system of simultaneous eigenvectors which, moreover, have the important additional property of being at the same time the eigenvalues. This last role can be specified for the different members of the hypergroup, by means of the (arbitrary) constants $\alpha_i$. Furthermore, equation (6) provides us with an explicit formula for the common eigenvector-eigenvalues $Y_{ij}$ in terms of the components $N_{ij}^k$ of the hypergroup, of the arbitrarily selected constants $\alpha_i$, and of a set of very general algebraic quantities, $e_i$, whose only restriction is to satisfy equation (5). It is important to notice this last point, in particular, the fact that the $e_i$ need not be related at all with (or by) the $N_{ij}^k$.

In the case of the very general solution (6) we are treating, it is clear that the $Y_{ij}$ are *not* symmetric: $Y_{ji} \neq Y_{ij}$, for $i \neq j$. We have, in particular,

$$Y_{0j} = \alpha_0 \sum_{lm} N_{jl}^m e^l \cdot e_m \equiv \alpha_0 c_j \qquad Y_{i0} = \alpha_i \sum_l e^l \cdot e_l \equiv d_i. \tag{8}$$

Notice also that putting $h=0$ in the eigenvalue equation (2), we get, in particular,

$$\sum_k N_{ij}^k c_k = c_i c_j \tag{9}$$

while setting instead, on the other hand, $i=0$ and using the last equation of (1), we obtain

$$Y_{hj} = \frac{1}{\alpha_h} Y_{h0} Y_{hj} \tag{10}$$

which yields a further constraint on the $e_i$, namely

$$\sum_i e^i \cdot e_i = 1. \tag{11}$$

That is, the $e_i$ must build a partition of the unity. This condition is easy to fulfill in terms of matrix products, as we shall see below (there is no conflict with the definition of the Kronecker symbol in equation (5), the product $\cdot$ being non-commutative and the 1 being the identity matrix). Consequently,

$$d_i = \alpha_i \qquad c_0 = 1 \qquad Y_{00} = \alpha_0. \tag{12}$$

A natural, unitary vector realization of the $e_i$ algebra in terms of matrix products, $e^i = e_i^T$, can be provided immediately.

A symmetric set of eigenvalue-eigenvectors $Y_{ij}$ is not difficult to construct, if one is willing to content oneself with the 'degenerate' situation

$$Y_{ij} = \alpha_0 c_i c_j = \alpha_0 \sum_{klm} N_{il}^k N_{jk}^m e^l \cdot e_m. \tag{13}$$

In fact, using equation (1) it is easy to prove that

$$\sum_l N_{ih}^l N_{jl}^k = \sum_l N_{jh}^l N_{il}^k. \tag{14}$$

That this is the form of the $Y_{ij}$ imposed by the symmetry condition $Y_{ji}=Y_{ij}$ can be checked by observing that this condition leads to the constraint

$$\alpha_i=\alpha_0\sum_{lm} N_{il}^m e^l\cdot e_m=\alpha_0 c_i. \tag{15}$$

These symmetric $Y_{ij}$ (13) satisfy equation (2) under the specific ordering

$$\sum_k N_{ij}^k Y_{hk}\alpha_h=Y_{hi}Y_{hj} \tag{16}$$

as can be checked directly, using the properties of the hypergroup repeatedly. Now, this situation corresponds, in this abstract general case, to the degenerate sectors which show up in the C* realization of the fusion-algebra hypergroup [3], i.e. to the case where Rehren's $Y_{ij}$ are given by

$$Y_{ij}=d_i d_j. \tag{17}$$

But with the very big difference that in our case the (operators) $c_i$ appear, instead of the constants $d_i$. The fundamental *lemma* concerning either the orthogonality or parallelism of the 'weight' vectors $Y_i$ makes use of equations (2) and of the symmetry $Y_{ji}=Y_{ij}$ only, and is also valid in the present (abstract) symmetric situation. It is fulfilled in the sense of the generalized parallelism $Y_{ij}\alpha_k=Y_{ik}\alpha_j$, since the $\alpha_j$s are *not* merely constants here.

One can use a different strategy and try to satisfy equations (2) by means of another set of eigenvector–eigenvalues, of the form

$$Y_{ij}=\sum_k N_{ij}^k\beta^i\cdot\beta_k^{-1}\cdot\beta^j\gamma_k \tag{18}$$

(or similar, e.g. with some—possibly all—of the $\beta^i$ absent); the product which appears in equation (18) is taken to be non-commutative, in general. If the $\beta^i$ and $\gamma_k$ are not functions of the $N_{ij}^k$, then it turns out that the only solutions, aside from the already known relation (6), that come out of this ansatz are

$$Y_{ij}=\sum_k N_{ij}^k u^j\cdot u_k\alpha_k \qquad u_i\cdot u^j=\frac{1}{\alpha_i}\delta_i^j \qquad u^i\cdot u_j=\frac{1}{\alpha_j}\delta_j^i \tag{19}$$

which looks very much like relation (6)—but notice that there is now no sum over $j$—and the most simple one

$$Y_{ij}=N_{ij}^j v_j \qquad v_i\cdot v^j=\delta_i^j \qquad v^i\cdot v_j=\delta_j^i v_j \qquad v_i\cdot v_j=\delta_{ij}v_j \tag{20}$$

which satisfies equations (2) with $\alpha_h=1$. A solution of the general type (18) is only possible if elaborated constraints involving the $N_{ij}^k$, $\beta^i$ and $\gamma_k$ are imposed. Precisely, one of the possibilities that one gets in this way corresponds to the physical realization considered by Rehren [3], namely

$$Y_{ij}=\sum_k N_{ij}^k\frac{\omega^i\omega^j}{\omega^k}d_k \tag{21}$$

where the $\omega_i$ are the statistic phases corresponding to the different inequivalent irreducible representations appearing in the classification of low-dimensional field theories, and the $N_{ij}^k$ are the multiplicities of each such representation in the tensor product of any two of them (see below).

## 3. The precise relation between Rehren's and Verlinde's algebras

From the arguments of the preceding section, it is clear that a symmetric solution—such as that of Rehren—is a very special case among the sets of eigenvectors which satisfy equations (2). We are going to see now that this fact actually reflects in the generation, via C* theory, of the modular-conformal algebra corresponding to the interpretation of the hypergroup as consisting of fusion matrices. In fact, let us consider the definition of $Y_{ij}$ corresponding to the C*QFT formulation [3]

$$Y_{ij} = \omega_i \omega_j \sum_k N_{ij}^k \frac{d_k}{\omega_k} \tag{22}$$

where here $N_{ij}^k$ denotes the multiplicity of $[\rho_k]$ (equivalence class of irreducible superselection sectors in $\Delta_0$, the set of transportable morphisms possessing conjugates and having finite statistics) in the class $[\rho_i\rho_j]$, and $d_i$ is the statistical dimension and $\omega_i$ the statistical phase corresponding to the equivalence class $[\rho_i]$, respectively. Remember that the statistics of a sector is a unitary operator $\varepsilon_\rho = \varepsilon(\rho, \rho)$ inducing a representation of the braid group (in the low-dimensional situation), and that $\varepsilon_\rho$ is a particular case of the unitary intertwiners $\varepsilon(\rho_i, \rho_j)$ which always exist from $\rho_i\rho_j$ to $\rho_j\rho_i$ given any two sectors $\rho_i$ and $\rho_j$.

Actually, in [3] the definition of $Y_{ij}$ is originally given through the (unique) left inverse $\phi_i$ of $\rho_i$ by

$$Y_{ij} = d_i d_j \phi(\varepsilon(\rho_j, \rho_i)^* \varepsilon(\rho_i, \rho_j)^*) \tag{23}$$

which in terms of the $N_{ij}^k$ yields the preceding expression (22) and, in particular,

$$Y_{ji} = Y_{ij} = Y_{ij}^* = Y_{\bar{i}\bar{j}} \tag{24}$$

and, contracting with $d_j/\omega_j$,

$$\sum_j Y_{ij} \frac{d_j}{\omega_j} = \sigma d_i \omega_i \tag{25}$$

where $\sigma = \Sigma_i d_i^2/\omega_i$, and satisfies $|\sigma|^2 = \Sigma_i d_i^2$. In this notation, the indices with a bar (such as $\bar{i}$) correspond to the representations $(\rho_{\bar{i}})$, which are conjugate to the irreducible ones $(\rho_i)$.

It has been proven by Rehren [3] that the matrices

$$S = \frac{1}{|\sigma|} Y \qquad T = \left(\frac{\sigma}{|\sigma|}\right)^{1/3} \operatorname{diag}(\omega_i) \qquad C = (\delta_{i\bar{j}}) \tag{26}$$

($C$ is the conjugation matrix), satisfy the algebra

$$S^2 = C \qquad TSTST = S \qquad TC = T \qquad SS^\dagger = TT^\dagger = I \tag{27}$$

of the modular-conformal transformations, first derived by Verlinde [4] and rigorously investigated in [5]†. Now, the essential point is the following. In the derivation of these relations, aside from the fact that the $Y_{ij}$ constitute a simultaneous set of eigenvalue-eigenvectors of the multiplicity hypergroup $N_{ij}^k$ (2), use is also made of the symmetry relations (24) and of the particular form of the expression (22), more

† For the derivation of the same algebra from the operator formalism of the Chern-Simons gauge theory with gauge group SU($N$) see [6], and for an extension of the standard procedure to currents of integer (rather than half-integer) spin see [7].

specifically, of (25). The algebra (27) is remarkably the same as that which appears in 2D conformal field theory, coming from the monodromy relations and modular invariance properties, when the $N_{ij}^k$ are the corresponding fusion matrices. But it is the case that in the C* formalism neither conformal nor modular invariance are postulated at any place, and everything comes out of the just described particular expressions for the $Y_{ij}$. As explained above, the relations (27) are not an outcome of the hypergroup properties alone (which turn out to be too weak in order to render them), and the question now is if the C*-algebraic expression for the $Y_{ij}$ is, on the contrary, too strong (i.e. restrictive), so that it yields *additional* relations to the algebra (27) above.

The answer to this question is indeed affirmative. It comes immediately out of the expressions (22) and (24) above. After some direct manipulations, we find the new algebraic relation

$$(ST^{\dagger}S)_{i0}=\left(\frac{\sigma}{|\sigma|}\right)^{1/3}(TS)_{i0} \tag{28}$$

which is actually a new algebraic constraint (of vector type) to the modular-conformal (sometimes called Verlinde) algebra (27). It is in fact independent of the relations in (27). With a little more work, another basic equation can be obtained, this time of matrix type, namely

$$\sum_k (TS)_{ik}S^{-1}_{k0}(S^{\dagger}T^{\dagger}S)_{k0}(ST)_{kj}=\left(\frac{\sigma}{|\sigma|}\right)^{1/3}S_{ij}. \tag{29}$$

Actually, this equation implies the preceding one (28), but in a very non-trivial way. A scalar relation which is derived by taking the trace of the last expression is the following:

$$\sum_i (ST^2S)_{ii}S^{-1}_{i0}(S^{\dagger}T^{\dagger}S)_{i0}=\left(\frac{\sigma}{|\sigma|}\right)^{1/3}\operatorname{Tr}S. \tag{30}$$

## 4. Conclusions

Summing up, the fact that the $Y_{ij}$ as defined by Rehren are symmetric and given by such particular expression (23) yields the new, additional restrictions (28)–(30), not present (in principle) in the 2D CFT fusion hypergroup. In particular, as explicitly stated in [5], where the rigorous proof of the Verlinde relations was carried out, the symmetry of the $S$ operator is *only pertinent when the fields are self-conjugate.* But it turns out that the symmetry of the $Y_{ij}$ (and hence of $S$) under interchange of the indices $i$ and $j$ is the basic ingredient in Rehren's derivation of the Verlinde algebra, and also (it looks likely) in any abstract derivation of the same, directly from the hypergroup properties.

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